$f : \mathbb{R}^d \to \mathbb{R} g N \{_{i,g}\} \subset \mathbb{R}^d$

$_{r_2} - _{r_3} f$

$$\begin{aligned} _{i,g} &= _{r_1} + F(_{r_2} - _{r_3}) \\ u_{i,j} &= v_{i,j} U(0,1) < CR j = j \\ _{i,g+1} &= _{i,g} f() \le f() \end{aligned}$$

$f N, g \to \infty$

F, CR

$$F_i^{g+1} = \begin{cases} F + r_1 F & r_2 < \tau_1, \\ F_i^g \end{cases} \qquad CR_i^{g+1} = \begin{cases} U(0,1) & r_3 < \tau_2, \\ CR_i^g \end{cases}$$

$\tau_1 = \tau_2 = 0.1 F 0.8$

$W = (W_t)_{t \ge 0} (\Omega, \mathcal{F}, \mathbb{P})$

$W_0 = 0$

$W_t - W_s \perp \mathcal{F}_s t > s$

$W_t - W_s \sim \mathcal{N}(0, t-s) 0 \le s < t$

$t \mapsto W_t(\omega)$

$\mathbb{E}[W_t] = 0 \quad (W_t) = t \quad (W_s, W_t) = (s, t)$

$[W]_T = T p p > 2$

$c^{-1/2} W_{ct} \overset{d}{=} W_t H = \frac{1}{2}$

$S_t = S_0\big((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\big)$

$f \in \mathcal{L}^2 \mathbb{E} \int_0^T f_t^2 \, dt < \infty$

$$\int_0^T f_t \, dW_t \;=\; L^2 \underset{|\pi| \to 0}{} \sum_k f_{t_k} (W_{t_{k+1}} - W_{t_k}).$$

$$\mathbb{E}\left[\left(\int_0^T f_t \, dW_t \right)^2 \right] = \mathbb{E} \int_0^T f_t^2 \, dt.$$

$dX_t = \mu_t\,dt + \sigma_t\,dW_t F \in C^{1,2}([0,T]\times\mathbb{R})$

$$dF(t,X_t) = \partial_t F\,dt + \partial_x F\,dX_t + \tfrac{1}{2}\partial_{xx}F\,\sigma_t^2\,dt.$$

$\frac{1}{2}\sigma^2\partial_{xx}F d[W]_t = dt$

$X_t = \ S_t dS_t = \mu S_t\,dt + \sigma S_t\,dW_t$

$$dX_t = \left(\mu - \tfrac{1}{2}\sigma^2\right)dt + \sigma\,dW_t.$$

$$dX_t = b(t,X_t)\,dt + \boldsymbol{\sigma}(t,X_t)\,dW_t, \quad X_0 = x_0.$$

$b, \boldsymbol{\sigma}\mathbb{E}[_{t\le T}\|X_t\|^2] < \infty$

$$\begin{aligned}
dX &= \sigma\,dW \\
dX &= \mu X\,dt + \sigma X\,dW \\
dX &= \kappa(\theta - X)\,dt + \sigma\,dW && \mathcal{N}(\theta,\,\sigma^2/2\kappa) \\
dX &= \kappa(\theta - X)\,dt + \sigma\sqrt{X}\,dW && (2\kappa\theta/\sigma^2,\,\sigma^2/2\kappa)
\end{aligned}$$

$$dX_t = \kappa(\theta - X_t)\,dt + \sigma\,dW_t.$$

$$X_t = \theta + (X_0 - \theta)e^{-\kappa t} + \sigma\int_0^t e^{-\kappa(t-s)}\,dW_s.$$

$\tau_{1/2} = \ 2/\kappa\kappa = 0.2 \approx 3.5$

Δt

$$\ell(\kappa,\theta,\sigma) = -\frac{1}{2}\sum_{i=1}^{n}\left[(2\pi\hat{\sigma}_i^2) + \frac{(X_{t_i} - \hat{\mu}_i)^2}{\hat{\sigma}_i^2}\right],$$

$\hat{\mu}_i = \theta + (X_{t_{i-1}} - \theta)e^{-\kappa\Delta t}\hat{\sigma}_i^2 = \dfrac{\sigma^2}{2\kappa}(1 - e^{-2\kappa\Delta t})$

$N = (N_t)_{t\ge 0}\lambda > 0$

$N_0 = 0$

$\mathbb{P}(N_{t+h} - N_t = 1) = \lambda h + o(h)\mathbb{P}(\Delta N > 1) = o(h)$

$N_t \sim (\lambda t)(\lambda)\tilde{N}_t = N_t - \lambda t$

$$\frac{dS_t}{S_{t^-}} = \mu\, dt + \sigma\, dW_t + d\left(\sum_{k=1}^{N_t}(e^{J_k}-1)\right),$$

$N_t \sim (\lambda) J_k \sim \mathcal{N}(\mu_J, \sigma_J^2)$

$$C = \sum_{n=0}^{\infty} \frac{e^{-\lambda' T}(\lambda' T)^n}{n!} \cdot C\big(S_0, K, T, r_n, \sigma_n^2\big)\,,$$

$$\lambda' = \lambda\, e^{\mu_J + \frac{1}{2}\sigma_J^2},$$

$$r_n = r - \lambda\big(e^{\mu_J+\frac{1}{2}\sigma_J^2} - 1\big) + \frac{n(\mu_J + \frac{1}{2}\sigma_J^2)}{T},$$

$$\sigma_n^2 = \sigma^2 + \frac{n\sigma_J^2}{T}.$$

$$\mathbb{E}[e^{i\xi X_t}] = \left(t\left[ib\xi - \tfrac{1}{2}\sigma^2\xi^2 + \int_{\mathbb{R}\setminus\{0\}}\big(e^{i\xi z} - 1 - i\xi z\,{}_{|z|\le 1}\big)\nu(dz)\right]\right),$$

$(b, \sigma^2, \nu)\nu \int (1 \wedge z^2)\nu(dz) < \infty$

ν

$\nu = 0$

$\nu(dz) \propto e^{-c|z|}/|z|$

$e^{-G|z|}/|z|^{1+Y} e^{-Mx}/x^{1+Y}$ $Y \in (0,2)$

α $c|z|^{-1-\alpha}$

$$dX_t = b(X_{t^-})\, dt + \sigma(X_{t^-})\, dW_t + \int_{\mathbb{R}} c(X_{t^-}, z)\, \widetilde{N}(dt, dz),$$

$\widetilde{N}(dt, dz) = N(dt, dz) - \nu(dz)\, dt$

$$dF(X_t) = \mathcal{L}F\, dt + \partial_x F\, \sigma\, dW_t + \int \big[F(X_{t^-} + c) - F(X_{t^-})\big]\widetilde{N}(dt, dz),$$

$$\mathcal{L}F = b\,\partial_x F + \tfrac{1}{2}\sigma^2 \partial_{xx} F + \int \big[F(x+c) - F(x) - c\,\partial_x F\big]\nu(dz).$$

u_t

$V(t,x)$
(X_t, p_t)

$X_t \in \mathbb{R}^d dX_t = b(X_t,u_t)\,dt + \sigma(X_t,u_t)\,dW_t$

$$J(t,x;u) = \mathbb{E}\left[\int_t^T \ell(X_s,u_s)\,ds + g(X_T)\,\middle|\, X_t = x\right].$$

$V(t,x) = {}_u\, J(t,x;u)$

$$\boxed{-\partial_t V = {}_{u\in\mathcal{U}}\Big[\underbrace{\ell(x,u)} + \underbrace{\nabla_x V^\top b(x,u)} + \underbrace{\tfrac{1}{2}\big(\sigma\sigma^\top(x,u)\,\nabla_x^2 V\big)}\Big], \quad V(T,\cdot) = g.}$$

$V u^\star(t,x) = {}_u[\ell(x,u) + \nabla_x V^\top b(x,u)]$

$X_t dX_t = (r + u_t(\mu - r))X_t\,dt + u_t\sigma X_t\,dW_t u_t \mathbb{E}[\,X_T]$
$V(t,x) = \,x + f(t)$

$$f'(t) = -r - \frac{(\mu-r)^2}{2\sigma^2}, \qquad f(T) = 0.$$

$$u^\star = \frac{\mu - r}{\sigma^2}.$$

$\ell = x^\top Q x + u^\top R u b = Ax + Bu V(t,x) = x^\top P(t) x + v(t) P$

$$-\dot{P} = A^\top P + PA - PBR^{-1}B^\top P + Q, \quad P(T) = Q_T.$$

$u_t^\star = -R^{-1}B^\top P(t) X_t$

$X_0 T X_t u_t < 0$

$$dX_t = u_t\,dt, \qquad \ell(x,u) = \alpha x^2 + \beta u^2.$$

$\sigma = 0 A = 0 B = 1 Q = \alpha R = \beta$

$$u^\star(t) = -\frac{\kappa}{(\kappa T)}\,(\kappa(T-t))\,X_0, \qquad \kappa = \sqrt{\alpha/\beta}.$$

$\kappa \to$

$\mathcal{H}(x,u,p) = \ell(x,u) + p^\top b(x,u)(X^\star, u^\star)p_t$

$$\dot p_t = -\nabla_x \mathcal{H}(X_t^\star, u_t^\star, p_t), \quad p_T = \nabla_x g(X_T^\star),$$

$u_t^\star = {}_u\, \mathcal{H}(X_t^\star, u, p_t)$

$p_t X_t$

$$p_t = \nabla_x V(t, X_t^\star) = \frac{\partial()}{\partial x}.$$

$dx p_t^\top dx$

${}_0 2.Integrate forward : dX = b(X, u*(X,p))dt[state ODE]3.Integrate backward : dp = -grad_x H(X, u*, p)dt[costate ODE]4.Check boundary : p_T = grad g(X_T)5.Not satisfied-> update p_0(Newton/gradient)-> back to 2$

$g(x) = -x p_T = -1/X_T^\star \dot p_t = -p_t(r + u^\star(\mu - r))$

$$p_t = -\frac{e^{-(T-t)(r+u^\star(\mu-r))}}{X_t^\star},$$

$p_t = \partial_x V = 1/x \partial_u \mathcal{H} = 0 u^\star = (\mu - r)/\sigma^2$

$(X_t^\star, p_t) T^\star \mathbb{R}^d$

$$-\partial_t V = {}_u\Big[\ell + \nabla V^\top b + \tfrac{1}{2}(\sigma\sigma^\top \nabla^2 V) + \underbrace{\int \big[V(x+c) - V(x) - \nabla V^\top c\big]\nu(dz)}\Big].$$

$cx x + c V(x+c) - V(x) \nabla V^\top c$

$$dX_t = u_t\,dt + d\left(\sum_{k=1}^{N_t} Z_k\right), \quad Z_k \sim \mathcal{N}(0, \sigma_J^2),\ N_t \sim (\lambda).$$

$V(t,x)$

$$-\partial_t V = {}_u\Big[\alpha x^2 + \beta u^2 + \partial_x V\, u + \lambda\, \mathbb{E}_z[V(x+z) - V(x) - z\,\partial_x V]\Big].$$

$\frac{1}{2}\lambda\sigma_J^2\,\partial_{xx} V \sigma^2 = \lambda \sigma_J^2$

$V C^{1,2}$

V

$V \phi V(t_0, x_0) V - \phi$

$$-\partial_t \phi(t_0, x_0) \le {}_u \Big[\ell(x_0, u) + \nabla_x \phi^\top b + \tfrac{1}{2}\,(\sigma\sigma^\top \nabla^2 \phi)\Big].$$

VV

$g(x) = (K - x)^+$

$$\Big(-\partial_t V - \mathcal{L}V,\ V - (K - x)^+\Big) = 0,$$

$\mathcal{L}V = rx\partial_x V + \frac{1}{2}\sigma^2 x^2 \partial_{xx} V - rV$

$V > (K - x)^+$

$V = (K - x)^+$

$\partial_x V \partial_{xx} V V \in C^1 V \notin C^2$

$\Delta t \le C\,(\Delta x)^2$

$$\begin{aligned} -\partial_t u - \nu \partial_{xx} u + H(x, \partial_x u, m) &= 0, & u(T, x) &= g(x), \\ \partial_t m - \nu \partial_{xx} m - \partial_x(m\,\partial_p H) &= 0, & m(0, x) &= m_0(x). \end{aligned}$$

Hm

$^0 = m_0(e.g.Gaussian)2.SolveHJBbackward- > u^{k+1}3.Extractoptimaldrift : alpha*(x,t) = -d_p H(x, d_x u^{k+1}, m^k)4.SolveFokker-Planckforwardwithalpha* - > m^{k+1}5.Check||m^{k+1} - m^k||_1 < eps; ifnot, k++- > goto2$

$\|m^{k+1} - m^k\|_1 \|u^{k+1} - u^k\|_\infty$

$$_t = F_{t-1} + {}_t, \quad {}_t \sim \mathcal{N}(0, Q); \qquad {}_t = H_t + {}_t, \quad {}_t \sim \mathcal{N}(0, R).$$

$$\hat{}^{-}_t = F_{t-1}, \qquad P_t^- = F P_{t-1} F^\top + Q.$$

$$
\begin{aligned}
K_t &= P_t^- H^\top (H P_t^- H^\top + R)^{-1},\\
\hat{}_t &= \tilde{}_t^- + K_t(_t - H\tilde{}_t^-),\\
P_t &= (I - K_t H) P_t^-.
\end{aligned}
$$

$K_t K \to 0 K \to H^{-1}$

$\hat{}_t = \mathbb{E}[_t \mid {}_{1:t}] D(p(_t|_{1:t}) \,\|\, \mathcal{N}(\hat{}_t, P_t))$

$d_t = A_t\, dt + B\, d_t d_t = C_t\, dt + d_t$

$$\dot{P} = AP + PA^\top + BQB^\top - PC^\top R^{-1} CP, \qquad P(0) = P_0.$$

$\pi(x) \propto e^{-U(x)} q(x' \mid x)$

$$\alpha(x \to x') = \left(1, \frac{\pi(x')q(x \mid x')}{\pi(x)q(x' \mid x)}\right).$$

$\pi(x)\alpha(x \to x') = \pi(x')\alpha(x' \to x)\pi$

$q(x'|x) = \mathcal{N}(x, h^2 I_d) h^\star \approx 2.38/\sqrt{d} \approx 2345$

$$x' = x - \tfrac{h^2}{2}\nabla U(x) + h\,\xi, \quad \xi \sim \mathcal{N}(0, I_d),$$

$$dX_t = -\nabla U(X_t)\, dt + \sqrt{2}\, dW_t,$$

$\pi \propto e^{-U}$

$O(d^{1/3}) O(d)$

$Z_t \in \{1, \dots, K\} A_{ij} = \mathbb{P}(Z_t = j \mid Z_{t-1} = i) Y_t \mid Z_t = k \sim B_k(y)$

$$
\begin{aligned}
\alpha_t(k) &= B_k(y_t) \sum_j \alpha_{t-1}(j) A_{jk}, & \beta_t(k) &= \sum_j A_{kj} B_j(y_{t+1}) \beta_{t+1}(j).\\
\gamma_t(k) &= \frac{\alpha_t(k)\beta_t(k)}{\sum_j \alpha_t(j)\beta_t(j)}, & \xi_t(j,k) &= \frac{\alpha_t(j) A_{jk} B_k(y_{t+1}) \beta_{t+1}(k)}{\mathcal{L}}.
\end{aligned}
$$

$$\hat{A}_{jk} = \frac{\sum_t \xi_t(j,k)}{\sum_t \gamma_t(j)}, \qquad \hat{\mu}_k = \frac{\sum_t \gamma_t(k)\, y_t}{\sum_t \gamma_t(k)}.$$

$\mathcal{L}(\theta)$

$\delta_t(k) = {}_j\, \delta_{t-1}(j) A_{jk} \cdot B_k(y_t) O(TK^2)$

---

p, q

$$D(p \,\|\, q) = \int p(x)\ \frac{p(x)}{q(x)}\, dx \ \geq\ 0,$$

$p = q$

$= 2k - 2\ \hat{\mathcal{L}} = k\ n - 2\ \hat{\mathcal{L}} D(p \,\|\, p_\theta)$

$p(x; \theta)$

$$\mathcal{I}(\theta)_{ij} = \mathbb{E}_{x \sim p}\Big[\partial_{\theta_i}\ p\ \partial_{\theta_j}\ p\Big] = -\mathbb{E}\Big[\partial^2_{\theta_i \theta_j}\ p\Big].$$

$\hat{\theta}(\hat{\theta}) \succeq \mathcal{I}(\theta)^{-1}$

$B_k = \mathcal{N}(\mu_k, \sigma_k^2) \mathcal{I}(\mu_k) = \sigma_k^{-2} \quad \mathcal{I}(\sigma_k^2) = (2\sigma_k^4)^{-1}$

$$I(X; Y) = D(p(X, Y) \,\|\, p(X)p(Y)) = H(X) - H(X \mid Y) \geq 0.$$

$$_{Y_i}\Big[I(Y_i;) - \frac{1}{|S|} \sum_{Y_j \in S} I(Y_i; Y_j)\Big].$$

$\nabla_\theta \mathcal{L} \mathcal{I}(\theta)^{-1} \nabla_\theta \mathcal{L}$

$(M, g) M g_p T_p M$

$$\ddot{\gamma}^k + \sum_{i,j} \Gamma^k_{ij}\, \dot{\gamma}^i \dot{\gamma}^j = 0,$$

$\Gamma^k_{ij} = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$

$\mathcal{M} = \{p(\cdot; \theta)\} g_{ij}(\theta) = \mathcal{I}(\theta)_{ij}$

$(\mathcal{M}, g)$

$$\theta \leftarrow \theta - \eta\, \mathcal{I}(\theta)^{-1} \nabla_\theta \mathcal{L}.$$

$$D(p_\theta \,\|\, p_{\theta+d\theta}) = \tfrac{1}{2}\, d\theta^\top \mathcal{I}(\theta)\, d\theta + O(\|d\theta\|^3),$$

$p(x;\theta) = h(x)\,(\theta^\top T(x) - A(\theta))em\eta = \nabla A(\theta) K = 0$

$G\mathfrak{g} = T_e G$

$SO(d) GL(d)$

G

$$\dot g(t) = g(t)\,\xi(t), \quad g \in G,\ \xi(t) \in \mathfrak{g}.$$

$(T^\star M, \omega)\omega = \sum_i dp_i \wedge dq_i \omega$

$(X_t^\star, p_t) T^\star \mathbb{R}^d$

ω

$K(\sigma)$

$K > 0$
$K = 0$
$K < 0$

$K = 0$

$W_t - W_s \sim \mathcal{N}(0, t-s)$
$dX = b\,dt + \sigma\,dW$
$N_t \sim (\lambda t)$
(b, σ^2, ν)
$-\partial_t V = {}_u[\ell + \nabla V^\top b + \frac{1}{2}\ \sigma\sigma^\top \nabla^2 V]$
$+ \int [V(\cdot + c) - V - \nabla V^\top c] \nu\, dz$
$\dot p = -\nabla_x \mathcal{H}\ u^\star = {}_u\,\mathcal{H}$
u, m
$K_t = P^- H^\top (HP^- H^\top + R)^{-1}$
$x' = x - \frac{h^2}{2} \nabla U + h\xi$

$\mathcal{I}_{ij} = \mathbb{E}[\partial_i \ell\, \partial_j \ell]$
$\mathcal{I}^{-1} \nabla_\theta \mathcal{L}$