

$$f:\mathbb{R}^d\rightarrow \mathbb{R} gN\{_{i,g}\}\subset \mathbb{R}^d$$

$$\frac{r_2-r_3}{\mathbb{R}^2}f$$

$$1,2,3$$

$$i r_1$$

$$_{ii}CR_i$$

$$_{ii}$$

$$\begin{array}{l} _{i,g}=r_1+F(r_2-r_3)\\ u_{i,j}=v_{i,j}U(0,1)<CRj=j\\ _{i,g+1}=_{i,g}f()\leq f() \end{array}$$

$$fN,g\rightarrow \infty$$

$$F,CR$$

$$F_i^{g+1}=\begin{cases} F+r_1F & r_2<\tau_1, \\ F_i^g \end{cases} \qquad CR_i^{g+1}=\begin{cases} U(0,1) & r_3<\tau_2, \\ CR_i^g \end{cases}$$

$$\tau_1=\tau_2=0.1F$$

$$f()=10d+\sum_i[x_i^2-10\,(2\pi x_i)]\!\approx\!10^df^*=0^*=$$

$$\partial_{x_i}f=2x_i+20\pi\left(2\pi x_i\right)$$

$$F(r_2-r_3)$$

$$d=10N=100\tau_1=\tau_2=0.1$$

f
 $FNCRN\approx 10dd>20$

$$F_i,CR_i[0.5,0.9]$$

$$n\xi_k=+1-1n1/\sqrt{n}$$

$$S^{(n)}_t=\frac{1}{\sqrt{n}}\sum_{k=1}^{[nt]}\xi_k.$$

$$n\rightarrow\infty S^{(n)}_t\overset{d}{\rightarrow}W_t\sim\mathcal{N}(0,t)$$

$$1/\sqrt{n}nn^{-1/2}n\cdot (1/\sqrt{n})^2\cdot t=t$$

$$W_t\sim \mathcal{N}(0,t)$$

$$\begin{array}{l} W=(W_t)_{t\geq 0}(\Omega,\mathcal{F},\mathbb{P})\\ W_0=0W_t-W_s\perp \mathcal{F}_st>sW_t-W_s\sim \mathcal{N}(0,t-s)0\leq s<tt\mapsto W_t(\omega) \end{array}$$

$$\begin{array}{l} \mathbb{E}[W_t]=0\\ (W_t)=t\\ (W_s,W_t)=(s,t)\\ \lim_{h\rightarrow 0}(W_{t+h}-W_t)/h\\ [W]_T=T\\ c^{-1/2}W_{ct}\stackrel{d}{=}W_tH=\tfrac{1}{2} \end{array}$$

$$[0,T]n\Delta=T/n$$

$$\sum_{k=0}^{n-1}(W_{t_{k+1}}-W_{t_k})^2\stackrel{?}{=}T\quad n\rightarrow\infty.$$

$$\begin{array}{l} (W_{t_{k+1}}-W_{t_k})^2\sim \Delta\cdot \chi_1^2\mathbb{E}[(W_{t_{k+1}}-W_{t_k})^2]=\Delta\\ \sum_{k=0}^{n-1}\Delta=n\Delta=T\\ (\sum(W_{t_{k+1}}-W_{t_k})^2)=n\cdot 2\Delta^2=2T^2/n\stackrel{n\rightarrow\infty}{\longrightarrow}0\\ \sum(W_{t_{k+1}}-W_{t_k})^2\stackrel{L^2}{\longrightarrow}TdW_t^2=dt \end{array}$$

$$\begin{array}{l} dx^2\approx dx\cdot dx\rightarrow 0(dW)^2=dt\\ \sim t^{0.5}1dt \end{array}$$

$$\propto \sqrt{t}$$

$$S_t = S_0\big((\mu - \tfrac{1}{2}\sigma^2)t + \sigma W_t\big)$$

$$\propto \sqrt{t}$$

$$S_t = S_0\big((\mu - \tfrac{1}{2}\sigma^2)t + \sigma W_t\big)$$

$$\int_0^T W_t\,dW_t W_{t_k}(W_{t_k} + W_{t_{k+1}})/2$$

W

$$f \in \mathcal{L}^2 \mathbb{E} \int_0^T f_t^2 \, dt < \infty$$

$$\int_0^T f_t \, dW_t \; := \; L^2_{|\pi| \rightarrow 0} \sum_k f_{t_k} (W_{t_{k+1}} - W_{t_k}).$$

$$\mathbb{E}\Big[\int_0^T f_t \, dW_t\Big] = 0 \mathbb{E}\Big[\Big(\int_0^T f_t \, dW_t\Big)^2\Big] = \mathbb{E} \int_0^T f_t^2 \, dt M_t = \int_0^t f_s \, dW_s \mathbb{E}[M_t \mid \mathcal{F}_s] = M_s$$

$$I_T = \sum_k f_{t_k} \Delta W_k$$

$$\mathbb{E}[I_T^2] = \sum_{j,k} \underbrace{\mathbb{E}[f_{t_j} \Delta W_j \cdot f_{t_k} \Delta W_k]}$$

$$j \neq k j < k f_{t_j} \Delta W_j f_{t_k} \mathcal{F}_{t_k} \Delta W_k \mathcal{F}_{t_k} = 0$$

$$j = k \mathbb{E}[f_{t_j}^2 (\Delta W_j)^2] = \mathbb{E}[f_{t_j}^2] \Delta t_j f_{t_j} \Delta W_j$$

$$\Rightarrow \mathbb{E}[I_T^2] = \sum_k \mathbb{E}[f_{t_k}^2] \Delta t_k \overset{|\pi| \rightarrow 0}{\longrightarrow} \mathbb{E} \int_0^T f_t^2 \, dt. \quad \checkmark$$

$$dX_t = \mu_t \, dt + \sigma_t \, dW_t F \in C^{1,2}([0,T] \times \mathbb{R})$$

$$dF(t,X_t) = \partial_t F \, dt + \partial_x F \, dX_t + \tfrac{1}{2} \sigma_t^2 \, \partial_{xx} F \, dt$$

$$dF = \underbrace{\left(\partial_t F + \mu_t \, \partial_x F + \tfrac{1}{2} \sigma_t^2 \, \partial_{xx} F\right)} \, dt + \underbrace{\sigma_t \, \partial_x F}_{\hspace{1.5cm}} \, dW_t.$$

$$F(t+dt,X_{t+dt})$$

$$dF = \partial_t F \, dt + \partial_x F \, dX + \tfrac{1}{2} \partial_{xx} F \, (dX)^2 + \underbrace{\partial_{tx} F \, dt \, dX}_{\rightarrow 0} + \dots$$

$$(dX)^2$$

$$dt dW_t$$

$$dt 0 0$$

$$dW_t 0 dt$$

$$\begin{aligned}
(dX_t)^2 &= (\mu_t dt + \sigma_t dW_t)^2 \\
&= \underbrace{\mu_t^2 (dt)^2}_0 + 2\mu_t \sigma_t \underbrace{dt \cdot dW_t}_0 + \sigma_t^2 \underbrace{(dW_t)^2}_{dt} \\
&= \sigma_t^2 dt.
\end{aligned}$$

$$dF = \partial_t F dt + \partial_x F (\mu_t dt + \sigma_t dW_t) + \frac{1}{2} \partial_{xx} F \cdot \sigma_t^2 dt$$

$$= (\partial_t F + \mu_t \partial_x F + \frac{1}{2} \sigma_t^2 \partial_{xx} F) dt + \sigma_t \partial_x F dW_t. \quad \checkmark$$

$$\begin{aligned}
&\frac{1}{2} \sigma^2 \partial_{xx} F dt \\
(dW)^2 &= 0(dW)^2 = dt
\end{aligned}$$

$$\in \mathbb{R}^n$$

$$dF = \partial_t F dt + \sum_i \partial_{x_i} F dX_i + \frac{1}{2} \sum_{i,j} \partial_{x_i x_j} F d[X_i, X_j]_t$$

$$d[X_i, X_j]_t = d\langle X_i, X_j \rangle_t$$

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

$$S_t$$

$$F(t,x) = x\partial_t F = 0\partial_x F = 1/x\partial_{xx} F = -1/x^2$$

$$d(S_t) = 0 + \frac{1}{S_t} dS_t + \frac{1}{2} \cdot \left(-\frac{1}{S_t^2}\right) \cdot \sigma^2 S_t^2 dt$$

$$= \frac{\mu S_t dt + \sigma S_t dW_t}{S_t} - \frac{1}{2} \sigma^2 dt$$

$$= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t.$$

$$S_T - S_0 = \left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma W_T.$$

$$\boxed{S_T = S_0 \left[\left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma W_T \right].}$$

$$-\tfrac{1}{2}\sigma^2T\mathbb{E}[S_T]=S_0+(\mu-\tfrac{1}{2}\sigma^2)T\mathbb{E}[S_T]=S_0e^{\mu T}e^{\mathbb{E}[X]}<\mathbb{E}[e^X]X$$

$$d(X_tY_t)$$

$$F(x,y)=xy$$

$$d(X_tY_t)=Y_t\,dX_t+X_t\,dY_t+d[X,Y]_t$$

$$d[X,Y]_t=\sigma_X\sigma_Y\,dtd(xy)=y\,dx+x\,dy(dx)^2=0$$

$$\int_0^TW_t\,dW_t=\tfrac{1}{2}W_T^2-\tfrac{1}{2}T.$$

$$\int_0^TW_t\,dW_t=\tfrac{1}{2}W_T^2-\tfrac{1}{2}T$$

$$F(t,x)=x^2/2dF=x\,dW+\tfrac{1}{2}\cdot 1\cdot dt=W_t\,dW_t+\tfrac{1}{2}\,dt\tfrac{1}{2}W_T^2-0=\int_0^TW_t\,dW_t+\tfrac{1}{2}T$$

$$\circ$$

$$+\tfrac{1}{2}\sigma^2\partial_{xx}F\\ f$$

$$\int f\circ dW=\int f\,dW+\tfrac{1}{2}\int \partial_xf\,\sigma\,dt$$

$$\rightarrow$$

$$\int_0^T f(X_t)\circ dW_t=\int_0^T f(X_t)\,dW_t+\tfrac{1}{2}\int_0^T f'(X_t)\sigma_t\,dt.$$

$$dX_t=b(t,X_t)\,dt+\boldsymbol{\sigma}(t,X_t)\,dW_t,\quad X_0=x_0.$$

$$\begin{aligned} b\sigma xt\|b(t,x)-b(t,y)\|+\|\sigma(t,x)-\sigma(t,y)\|&\leq L\|x-y\|\\ \|b(t,x)\|^2+\|\sigma(t,x)\|^2&\leq C^2(1+\|x\|^2)\\ \mathbb{E}\big[_{t\leq T}\|X_t\|^2\big]&<\infty\end{aligned}$$

$$X_t^{(0)}=x_0n\geq 0$$

$$X_t^{(n+1)}=x_0+\int_0^tb(s,X_s^{(n)})\,ds+\int_0^t\sigma(s,X_s^{(n)})\,dW_s.$$

$$\varepsilon_n(t)=\mathbb{E}\Big[_{s\leq t}|X_s^{(n+1)}-X_s^{(n)}|^2\Big] \\ L^2$$

$$\varepsilon_{n+1}(t)\leq 2(L^2T+L^2)\int_0^t\varepsilon_n(s)\,ds.$$

$$\varepsilon_n(t)\leq C\cdot \frac{(2L^2(T+1)t)^n}{n!}\rightarrow 0X^{(n)}L^2$$

$$X_tp(t,x)p$$

$$\frac{\partial p}{\partial t}=-\frac{\partial}{\partial x}[b(t,x)\,p]+\frac{1}{2}\frac{\partial^2}{\partial x^2}[\sigma^2(t,x)\,p].$$

$$\phi$$

$$\frac{d}{dt}\mathbb{E}[\phi(X_t)]=\mathbb{E}[\mathcal{L}\phi(X_t)]=\mathbb{E}\big[b\,\phi'+\tfrac{1}{2}\sigma^2\phi''\big]$$

$$\phi(X_t)x\phi p$$

$$b = \kappa(\theta - x)\sigma p_\infty(x) = \mathcal{N}(\theta, \sigma^2/2\kappa)$$

$$X_t$$

$$\begin{aligned}dX &= \sigma \, dW X_0 + \sigma W_t \\dX &= \mu X \, dt + \sigma X \, dW X_0 e^{(\mu - \sigma^2/2)t + \sigma W_t} \\dX &= \kappa(\theta - X) \, dt + \sigma \, dW \mathcal{N}(\theta, \sigma^2/2\kappa) \\dX &= \kappa(\theta - X) \, dt + \sigma \sqrt{X} \, dW (2\kappa\theta/\sigma^2, \sigma^2/2\kappa) \\dF &= \sigma F^\beta dW^1 d\sigma = \nu \sigma \, dW^2\end{aligned}$$

$$\Delta t$$

$$X_{t+\Delta t} \approx X_t + b(t,X_t)\,\Delta t + \sigma(t,X_t)\,\Delta W_t$$

$$\Delta W_t = \sqrt{\Delta t} \, Z Z \sim \mathcal{N}(0,1)$$

$$X_{t+\Delta t} \approx X_t + b \, \Delta t + \sigma \, \Delta W_t + \tfrac{1}{2} \sigma \, \sigma_x \big[(\Delta W_t)^2 - \Delta t \big].$$

$$\tfrac{1}{2} \sigma \sigma_x \big[(\Delta W_t)^2 - \Delta t \big] \sigma(X_t) dW_t$$

$$dX_t = \kappa(\theta - X_t) \, dt + \sigma \, dW_t.$$

$$X_t \theta$$

$$dX_t + \kappa X_t \, dt = \kappa \theta \, dt + \sigma \, dW_t.$$

$$e^{\kappa t}$$

$$d\big(e^{\kappa t} X_t\big) = e^{\kappa t} dX_t + \kappa e^{\kappa t} X_t \, dt = e^{\kappa t} \kappa \theta \, dt + e^{\kappa t} \sigma \, dW_t.$$

$$d(e^{\kappa t} X_t) = e^{\kappa t} dX_t + X_t \cdot \kappa e^{\kappa t} dt e^{\kappa t}$$

$$0t$$

$$e^{\kappa t} X_t - X_0 = \kappa \theta \int_0^t e^{\kappa s} \, ds + \sigma \int_0^t e^{\kappa s} \, dW_s$$

$$e^{\kappa t} X_t - X_0 = \theta (e^{\kappa t} - 1) + \sigma \int_0^t e^{\kappa s} \, dW_s.$$

$$e^{\kappa t}$$

$$X_t = \theta + (X_0 - \theta)e^{-\kappa t} + \sigma \int_0^t e^{-\kappa(t-s)} \, dW_s.$$

$$\theta$$

$$(X_0 - \theta)e^{-\kappa t} \kappa \\ \sigma \int_0^t e^{-\kappa(t-s)} \, dW_s$$

$$I_t = \sigma \int_0^t e^{-\kappa(t-s)} \, dW_s$$

$$\mathbb{E}[I_t] = 0, \qquad (I_t) = \sigma^2 \int_0^t e^{-2\kappa(t-s)} \, ds = \frac{\sigma^2}{2\kappa} (1 - e^{-2\kappa t}).$$

$$X_t \sim \mathcal{N}\bigg(\theta + (X_0 - \theta)e^{-\kappa t}, \, \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t})\bigg).$$

$$t \rightarrow \infty X_t \rightarrow \mathcal{N}(\theta, \sigma^2/2\kappa)$$

$$X_s$$

$$X_t \mid X_s \sim \mathcal{N}\bigg(\theta + (X_s - \theta)e^{-\kappa(t-s)}, \, \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa(t-s)})\bigg), \quad t > s.$$

$$\hat{\mu}(\tau) = \theta + (X_s - \theta)e^{-\kappa\tau}, \qquad \hat{\sigma}^2(\tau) = \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa\tau}), \quad \tau = t - s.$$

$$needlongseries to identify mean-reversionspeed)$$

$$\hat{\kappa}\hat{\theta}\hat{\sigma}^{0.5}$$

$$r_i = (X_{t_i} - \hat{\mu}_i)/\hat{\sigma}\mathcal{N}(0,1)$$

$$_i histogram vs N(0,1)$$

$$_i-2+2$$

$$r_i$$

$$|X_t-\hat{\theta}|>2\hat{\sigma}_\infty\hat{\sigma}_\infty=\hat{\sigma}/\sqrt{2\hat{\kappa}}X_t=\hat{\theta}\approx\hat{\tau}_{1/2}=2/\hat{\kappa}\approx4.6$$

$$\approx 2\hat{\sigma}_\infty=2\hat{\sigma}/\sqrt{2\hat{\kappa}}<2\hat{\sigma}_\infty$$

$$N=(N_t)_{t\geq 0}\lambda>0$$

$$N_0=0\mathbb{P}(N_{t+h}-N_t=1)=\lambda h+o(h)\mathbb{P}(\Delta N>1)=o(h)$$

$$N_t\sim (\lambda t)(\lambda)\tilde{N}_t=N_t-\lambda t$$

$$\lambda=2$$

$$\frac{dS_t}{S_{t-}}=\mu\,dt+\sigma\,dW_t+d\Big(\sum_{k=1}^{N_t}(e^{J_k}-1)\Big),$$

$$N_t\lambda J_k\sim \mathcal{N}(\mu_J,\sigma_J^2)$$

$$C=\sum_{n=0}^{\infty}\frac{e^{-\lambda'T}(\lambda'T)^n}{n!}\cdot C(S_0,K,T,r_n,\sigma_n^2)\,,$$

$$\lambda'=\lambda e^{\mu_J+\frac{1}{2}\sigma_J^2}r_n=r-\lambda(e^{\mu_J+\frac{1}{2}\sigma_J^2}-1)+n(\mu_J+\frac{1}{2}\sigma_J^2)/T\sigma_n^2=\sigma^2+n\sigma_J^2/T$$

$$ne^{-\lambda'T}(\lambda'T)^n/n!r_n\sigma^2T+n\sigma_J^2n$$

$$\sigma, \lambda, \mu_J, \sigma_J$$

$$\sigma \lambda \mu_J \sigma_J$$

$$KT\sum_{K,T}(C(K,T;\theta)-C)^2 evolution(DEisideal4params,non-convexlandscape).3.*$$

$$*Diagnostic:**PlotMertonvsmarket smile; expect fit within 0.5vega.$$

$$\mu_J$$

$$\mathbb{E}[e^{i\xi X_t}]=\Big(t\Big[ib\xi-\tfrac{1}{2}\sigma^2\xi^2+\int_{\mathbb{R}\setminus\{0\}}(e^{i\xi z}-1-i\xi z_{|z|\leq 1})\nu(dz)\Big]\Big)$$

$$(b,\sigma^2,\nu)\nu\int(1\wedge z^2)\nu(dz)<\infty$$

$$\nu$$

$$\nu=0$$

$$\nu(dz) \propto e^{-c|z|}/|z|$$

$$Y\in(0,2)$$

$$\alpha c|z|^{-1-\alpha}$$

$$dX_t=b(X_{t-})\,dt+\sigma(X_{t-})\,dW_t+\int_{\mathbb{R}}c(X_{t-},z)\,\tilde{N}(dt,dz),$$

$$\tilde{N}(dt,dz)=N(dt,dz)-\nu(dz)\,dt$$

$$dF(X_t)=\mathcal{L}F\,dt+\partial_xF\,\sigma\,dW_t+\int\big[F(X_{t-}+c)-F(X_{t-})\big]\tilde{N}(dt,dz),$$

$$\mathcal{L}F=b\,\partial_xF+\tfrac{1}{2}\sigma^2\partial_{xx}F+\int\big[F(x+c)-F(x)-c\,\partial_xF\big]\nu(dz).$$

$$u_t$$

$$V(t,x)\\(X_t,p_t)$$

$$X_t\in\mathbb{R}^d$$

$$dX_t=b(X_t,u_t)\,dt+\sigma(X_t,u_t)\,dW_t,$$

$$J(t,x;u)=\mathbb{E}\bigg[\int_t^T\ell(X_s,u_s)\,ds+g(X_T)\,\Big|\,X_t=x\bigg]\,.$$

$$V(t,x) = {}_uJ(t,x;u)$$

$$-\partial_t V = {}_{u\in\mathcal{U}}\bigg[\ell(x,u) + \nabla_x V^\top b(x,u) + \tfrac{1}{2}\left(\sigma\sigma^\top(x,u)\nabla_x^2 V\right)\bigg], \quad V(T,\cdot) = g.$$

$$\ell(x,u)\nabla_xV^\top b\frac{1}{2}\left(\sigma\sigma^\top\nabla^2V\right)$$

$$Vu^\star(t,x) = {}_u[\ell(x,u) + \nabla_x V^\top b(x,u)].$$

$$\begin{array}{l} X_t dX_t = (r + u_t(\mu - r))X_t \, dt + u_t \sigma X_t \, dW_t u_t \in \mathbb{R} \\ -\mathbb{E}[\, X_T] \\ V(t,x) = \, x + f(t) \end{array}$$

$$f'(t)=-r-\frac{(\mu-r)^2}{2\sigma^2},\qquad f(T)=0.$$

$$u^\star=\frac{\mu-r}{\sigma^2}\quad ({}).$$

$$\ell = x^\top Q x + u^\top R u b = Ax + Bu \sigma V(t,x) = x^\top P(t) x + v(t) P$$

$$-\dot{P}=A^\top P+PA-PBR^{-1}B^\top P+Q,\quad P(T)=Q_T.$$

$$u_t^\star = -R^{-1}B^\top P(t)X_t$$

$$X_0TX_tu_t<0$$

$$dX_t = u_t \, dt, \quad \ell(x,u) = \underbrace{\alpha x^2} + \underbrace{\beta u^2} \,.$$

$$A=0B=1Q=\alpha R=\beta$$

$$u^\star(t,x)=-\frac{\alpha}{\beta}\cdot\frac{(\kappa(T-t))}{(\kappa T)}\cdot X_0,\quad \kappa=\sqrt{\alpha/\beta}.$$

$$\kappa$$

$$(X_t,p_t)$$

$$\mathcal{H}(x,u,p)=\ell(x,u)+p^\top b(x,u)(X^\star,u^\star)p_t$$

$$\dot{p}_t=-\nabla_x\mathcal{H}(X_t^\star,u_t^\star,p_t),\quad p_T=\nabla_xg(X_T^\star),$$

$$u_t^\star=\mathop{u}\mathcal{H}(X_t^\star,u,p_t)$$

$$p_tX_t$$

$$p_t=\nabla_xV(t,X_t^\star)=\frac{\partial ()}{\partial x}.$$

$$\begin{array}{l} \ell=0g(x)=-\,xb=(r+u(\mu-r))x\mathcal{H}(x,u,p)=p(r+u(\mu-r))x\\ \dot{p}_t=-\partial_x\mathcal{H}=-p_t(r+u^\star(\mu-r))p_T=-1/X_T^\star\\ \partial_u\mathcal{H}=0u^\star=(\mu-r)/\sigma^2\\ p_t=-e^{-(T-t)(r+(\mu-r)u^\star)}/X_t^\star \end{array}$$

$$(X_t^\star,p_t)T^\star\mathbb{R}^d$$

$$-\partial_tV=\mathop{u}\Big[\ell+\nabla V^\top b+\tfrac12\big(\sigma\sigma^\top\nabla^2V\big)+\underbrace{\int\big[V(x+c(x,u,z))-V(x)-\nabla V^\top c(x,u,z)\big]\nu(dz)}\Big].$$

$$c x x + c V(x+c) - V(x) \nabla V^\top c$$

$$dX_t = u_t\,dt + \Delta J_t, \quad \Delta J_t \sim (\lambda, \mathcal{N}(0, \sigma_J^2)).$$

$$\sigma^2 = \lambda \sigma_J^2$$

$$VC^{1,2}$$

$$V\phi V(t_0,x_0)$$

$$-\partial_t\phi(t_0,x_0)\leq_u\Big[\ell(x_0,u)+\nabla_x\phi^\top b+\tfrac12\,(\sigma\sigma^\top\nabla^2\phi)\Big].$$

$$VV$$

$$\Delta t \leq C\,(\Delta x)^2$$

$$g(x) = (K-x)^+$$

$$\left(-\partial_t V - \mathcal{L}V, \, V - (K-x)^+\right) = 0.$$

$$V>gV=g$$

$$\partial_x V \partial_{xx} V V C^1 C^2$$

$$-\partial_t u - \nu \partial_{xx} u + H(x,\partial_x u,m) = 0, \quad u(T,x) = g(x),$$

$$\partial_t m - \nu \partial_{xx} m - \partial_x (m\,\partial_p H) = 0, \qquad m(0,x) = m_0(x).$$

$$Hm$$

$$\|m^{k+1}-m^k\|_1\|u^{k+1}-u^k\|_\infty$$

$$N\gg 1x_tT\bar{u}_t=\int u\,m(t,dx)$$

$$H(x,p,m) = {}_u\Big[\alpha x^2 + \beta u^2 + pu\Big] + \underbrace{\gamma \bar{u}(m)}\,x.$$

$$_t=F_{t-1}+_t,\;_t\sim\mathcal{N}(0,Q);\qquad _t=H_t+_t,\;_t\sim\mathcal{N}(0,R).$$

$$\hat{\varphi}_t^-=F_{t-1},\quad P_t^-=FP_{t-1}F^\top+Q.$$

$$K_t=P_t^-H^\top(HP_t^-H^\top+R)^{-1},\quad \hat{t}=\hat{\varphi}_t^-+K_t(t-H_t^-),\quad P_t=(I-K_tH)P_t^-.$$

$$K_tK\rightarrow 0K\rightarrow H^{-1}$$

$$P_t \rightarrow P_\infty(F,H)$$

$$\hat{t} = \mathbb{E}[_t \mid _{1:t}] D(p(t|_{1:t}) \parallel \mathcal{N}(\hat{t}, P_t))$$

$$d_t = A_t \, dt + B \, d_t d_t = C_t \, dt + d_t$$

$$\dot{P} = AP + PA^\top + BQB^\top - PC^\top R^{-1}CP, \qquad P(0) = P_0.$$

$$\begin{array}{l} x_t = 0.95x_{t-1} + w_tQ = 0.01y_t = x_t + v_tR = 1.0 \\ P_\infty \approx 0.17K_\infty \approx 0.15 \\ {}_t1.0|*0.8| \\ 0.6| \\ 0.4| \\ {}_0.2| \\ \\ R/Q = 100^{P_inf0.17+----->t(fastconvergentosteady-stateuncertainty)} \end{array}$$

$$\pi(x) \propto e^{-U(x)}q(x' \mid x)$$

$$\alpha(x\rightarrow x')=\Big(1,\frac{\pi(x')q(x\mid x')}{\pi(x)q(x'\mid x)}\Big).$$

$$\pi(x)\alpha(x\rightarrow x')=\pi(x')\alpha(x'\rightarrow x)\pi$$

$$h^\star \approx 2.38/\sqrt{d}$$

$$x' = x - \frac{h^2}{2} \nabla U(x) + h \, \xi, \quad \xi \sim \mathcal{N}(0, I_d),$$

$$dX_t = -\nabla U(X_t) \, dt + \sqrt{2} \, dW_t,$$

$$\pi \propto e^{-U}$$

$$O(d^{1/3})O(d)$$

$$\begin{aligned} (\kappa, \theta, \sigma) &\kappa \sim (2, 0.1) \theta \sim \mathcal{N}(0, 1) \sigma \sim (0.5) \\ d &= 3h = 0.02 \end{aligned}$$

$$Z_t \in \{1,\dots,K\} A_{ij} = \mathbb{P}(Z_t = j \mid Z_{t-1} = i) Y_t \mid Z_t = k \sim B_k(y)$$

$$K=3$$

$$\alpha_t(k) = B_k(y_t) \sum_j \alpha_{t-1}(j) A_{jk}, \qquad \beta_t(k) = \sum_j A_{kj} B_j(y_{t+1}) \beta_{t+1}(j).$$

$$\gamma_t(k) = \frac{\alpha_t(k)\beta_t(k)}{\sum_j \alpha_t(j)\beta_t(j)}, \qquad \xi_t(j,k) = \frac{\alpha_t(j)A_{jk}B_k(y_{t+1})\beta_{t+1}(k)}{\mathcal{L}}.$$

$$\hat{A}_{jk} = \frac{\sum_t \xi_t(j,k)}{\sum_t \gamma_t(j)}, \qquad \hat{\mu}_k = \frac{\sum_t \gamma_t(k) \, y_t}{\sum_t \gamma_t(k)}.$$

$$\mathcal{L}(\theta)$$

$$K=3T=4$$

$$\delta_t(k) = {}_j\delta_{t-1}(j)A_{jk}\cdot B_k(y_t)O(TK^2)$$

$$T=5820$$

$$K=3_{\mathfrak{b}aum_welch()}with20randomrestarts.$$

$$\gamma_t(k)$$

$$^{d_{ot-combustGFCCOVIDcrash}}$$

$$\gamma_{tevolutiontoswitchrisk-aversion}\alpha$$

$$p,q$$

$$D(p\parallel q)=\int p(x)\,\frac{p(x)}{q(x)}\,dx\;\geq\;0,$$

$$p=q$$

$$=2k-2\,\hat{\mathcal{L}}=k\,n-2\,\hat{\mathcal{L}}D(p\parallel p_\theta)$$

$$p(x;\theta)$$

$$\mathcal{I}(\theta)_{ij} = \mathbb{E}_{x \sim p} \big[\partial_{\theta_i} \, p \, \partial_{\theta_j} \, p \big] = - \mathbb{E} \Big[\partial_{\theta_i \theta_j}^2 \, p \Big] \, .$$

$$\hat{\theta}(\hat{\theta}) \succeq \mathcal{I}(\theta)^{-1}$$

$$B_k = \mathcal{N}(\mu_k, \sigma_k^2) \mathcal{I}(\mu_k) = \sigma_k^{-2} \mathcal{I}(\sigma_k^2) = (2\sigma_k^4)^{-1}$$

$$I(X;Y) = D\big(p(X,Y) \parallel p(X)p(Y)\big) = H(X) - H(X \mid Y) \geq 0.$$

$$I(X;Y)YXX \perp Y \Rightarrow I = 0YX \Rightarrow I = H(X)$$

$$_{Y_i}\Big[I(Y_i;)-\frac{1}{|S|}\sum_{Y_j\in S}I(Y_i;Y_j)\Big].$$

$$t$$

$$_t = (\gamma_t(1), \gamma_t(2), \gamma_t(3)).$$

$$H_t = -\sum_k \gamma_t(k) \; \gamma_t(k) \in [0,\,3]$$

$$H_t$$

$$3 \approx 1.10 H_t < 0.7$$

$$\nabla_{\theta} \mathcal{L} \mathcal{I}(\theta)^{-1} \nabla_{\theta} \mathcal{L}$$

$(M,g)Mg_pT_pM$

$$p$$

$$\ddot{\gamma}^k + \sum_{i,j} \Gamma_{ij}^k \dot{\gamma}^i \dot{\gamma}^j = 0,$$

$$\Gamma_{ij}^k=\tfrac{1}{2}g^{kl}(\partial_i g_{jl}+\partial_j g_{il}-\partial_l g_{ij})$$

$$\mathcal{M}=\{p(\cdot;\theta)\}g_{ij}(\theta)=\mathcal{I}(\theta)_{ij}$$

$$\theta \leftarrow \theta - \eta \mathcal{I}(\theta)^{-1} \nabla_{\theta} \mathcal{L}.$$

$$D(p_\theta \parallel p_{\theta+d\theta}) = \tfrac{1}{2} \, d\theta^\top \mathcal{I}(\theta) \, d\theta + O(\|d\theta\|^3)$$

$$p(x;\theta)=h(x)\left(\theta^\top T(x)-A(\theta)\right)K=0$$

$$p(x;\theta) = \mathcal{N}(\mu,\sigma^2)\theta = (\mu,\sigma^2)$$

$$\mathcal{I}(\theta)=\begin{pmatrix} 1/\sigma^2 & 0 \\ 0 & 1/(2\sigma^4) \end{pmatrix}.$$

$$\mathcal{L}=-\,p(x;\theta)$$

$$\tilde{\nabla}_{\theta}\mathcal{L}=\mathcal{I}^{-1}\nabla\mathcal{L}=\begin{pmatrix}\mu-x\\ \sigma^2-(x-\mu)^2/2\end{pmatrix}.$$

$$\mathcal{I}K=0$$

$$G\mathfrak{g}=T_eG$$

$$G$$

$$\dot{g}(t)=g(t)\,\xi(t),\quad g\in G,\,\xi(t)\in\mathfrak{g}.$$

$$(T^{\star}M,\omega)\omega=\sum_i dp_i\wedge dq_i\omega$$

$$(X_t^\star,p_t)T^\star\mathbb{R}^d$$

$$\omega$$

$$K(\sigma)$$

$$K=0$$

$$\begin{aligned} W_t-W_s&\sim\mathcal{N}(0,t-s)\\ dX&=b\,dt+\sigma\,dW\\ N_t&\sim(\lambda t)\\ (b,\sigma^2,\nu)&\\ -\partial_tV&=_u[\ell+\nabla V^\top b+\tfrac12\,\sigma\sigma^\top\nabla^2V]\\ &+\int[V(\cdot+c)-V-\nabla V^\top c]\nu\,dz\\ \dot p&=-\nabla_x\mathcal{H}u^\star=_u\mathcal{H}\\ u,m&\\ K_t&=P^-H^\top(HP^-H^\top+R)^{-1}\\ x'&=x-\tfrac{h^2}{2}\nabla U+h\xi\end{aligned}$$

$$\begin{aligned}\mathcal{I}_{ij}&=\mathbb{E}[\partial_i\ell\,\partial_j\ell] \\ \mathcal{I}^{-1}\nabla_{\theta}\mathcal{L}\end{aligned}$$
